Differential Geometry III: Riemannian Geometry

10.1 Let (S, g) be a Riemannian surface (i.e. $dimS = 2$). Suppose that, in polar coordinates (r, θ) around a point $p \in S$, the metric g takes the form

$$g = dr^2 + \big(f(r, \theta)\big)^2 d\theta^2$$

(recall that, as we showed in class, $\lim_{r\to 0} f(r, \theta) = 0$ and $\lim_{r\to 0} \partial_r f(r, \theta) = 1$).

(a) Show that the sectional curvature K of (S, g) satisfies at any point in this coordinate chart:

$$\frac{\partial^2 f}{\partial r^2} + Kf = 0.$$

(b) Derive an expression in polar coordinates for any metric of constant sectional curvature in dimension 2.

(c) Show that any two Riemannian surfaces with constant sectional curvature of the same value are locally isometric. Are they also globally isometric?

Solution. (a) We need to derive an expression for the sectional curvature K in the (r, θ) coordinate system. Note that the matrix of the metric components and its inverse take the form

$$[g] = \begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f^2 \end{bmatrix}, \quad [g]^{-1} = \begin{bmatrix} g^{rr} & g^{r\theta} \\ g^{\theta r} & g^{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f^{-2} \end{bmatrix}.$$

Therefore, using the formula $\Gamma^k_{ij} = \frac{1}{2} g^{kl}\big(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\big)$ for the Christoffel symbols, we can readily compute

$$\Gamma^r_{rr} = 0, \quad ,\Gamma^r_{r\theta} = 0, \quad \Gamma^r_{\theta\theta} = -f\partial_r f,$$
$$\Gamma^\theta_{rr} = 0, \quad \Gamma^\theta_{r\theta} = \frac{\partial_r f}{f}, \quad \Gamma^\theta_{\theta\theta} = \frac{\partial_\theta f}{2f}.$$

The formula ${R^a}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ck}\Gamma^k_{bd} - \Gamma^a_{dk}\Gamma^k_{bc}$ for the Riemann curvature tensor then yields:

$$\begin{aligned} R_{r\theta r\theta} &= g_{rr}{R^r}_{\theta r\theta} + g_{r\theta}{R^\theta}_{\theta r\theta} \\ &= \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^r_{\theta r} + \Gamma^r_{ra}\Gamma^a_{\theta\theta} - \Gamma^r_{\theta a}\Gamma^a_{\theta r} + 0 \\ &= \partial_r\big(-f\partial_r f\big) - 0 + 0 - (-f\partial_r f)\frac{\partial_r f}{f} \\ &= -f\partial_r^2 f. \end{aligned}$$

Therefore, since the sectional curvature K was defined by

$$K = \frac{R_{r\theta r\theta}}{\|\partial_r\|^2\|\partial_\theta\|^2 - \langle \partial_r, \partial_\theta\rangle^2},$$

we infer that

$$K = -\frac{\partial_r^2 f}{f}.$$

(b) In the case when $K = const$, integrating the relation

$$\partial_r^2 f + Kf = 0$$

with respect to r and using at $r = 0$ the boundary conditions $f|_{r=0} = 0$ and $\partial_r f|_{r=0} = 1$, we obtain the following explicit representation for $f(r, \theta)$:

1. In the case when $K > 0$:

$$f(r,\theta) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) \quad \text{and} \quad g = dr^2 + \frac{1}{K}\sin^2(\sqrt{K}r)d\theta^2$$

   (this is the round metric on the sphere of radius $\frac{1}{\sqrt{K}}$).

2. In the case when $K = 0$:

$$f(r,\theta) = r \quad \text{and} \quad g = dr^2 + r^2 d\theta^2$$

   (this is the flat metric g_E).

3. In the case when $K < 0$:

$$f(r,\theta) = \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}r) \quad \text{and} \quad g = dr^2 + \frac{1}{-K}\sinh^2(\sqrt{-K}r)d\theta^2$$

   (this is a rescaling of the hyperbolic metric by the factor $\frac{1}{\sqrt{-K}}$).

(c) From part (b) of this exercise, we deduce that if (S_1, g_1) and (S_2, g_2) have both constant curvature equal to K, then for every $p_1 \in S_1$ and $p_2 \in S_2$, in any local neighborhoods of those points covered by polar coordinates, the expression of the metric is the same. Thus, there is a neighborhood $\mathcal{U}_1$ of p_1 in S_1 and $\mathcal{U}_2$ of p_2 in S_2 such that $(\mathcal{U}_1, g_1)$ and $(\mathcal{U}_2, g_2)$ are isometric. These surfaces are not necessarily globally isometric; for instance, $(\mathbb{R}^2, g_E)$ and $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, g_E)$ have both vanishing sectional curvature, but the latter is compact while the former is not. Similarly, the round sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$ and the projective plane $(\mathbb{R}P^2, g_{P^2})$ are locally isometric with sectional curvature equal to $+1$, but are not globally isometric, as seen in the 2^{nd} exercise sheet.

10.2 (a) Let $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be an isometry. Show that, for any $X, Y, Z, W \in \Gamma(\mathcal{M})$ and any $p \in \mathcal{M}$,

$$R_h(F_*X, F_*Y, F_*Z, F_*W)|_{F(p)} = R_g(X, Y, Z, W)|_p,$$

where R_g, R_h are the Riemann curvature tensors associated to g, h, respectively, and $F_*(V) \doteq dF(V)$. *Hint: Use the fact that, for any such isometry F, $\nabla^{(h)}_{F_*X}(F_*Y) = F_*\big(\nabla^{(g)}_X Y\big)$.*

(b) Let $(\mathcal{M}, g)$ have the property that, for any $p, q \in \mathcal{M}$, any non-collinear $V_1, V_2 \in T_p\mathcal{M}$ and non-collinear $W_1, W_2 \in T_q\mathcal{M}$, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that $F(p) = q$ and the plane spanned by $\{F_*V_1, F_*V_2\}$ is the same as for $\{W_1, W_2\}$. Show that the sectional curvature is constant on $\mathcal{M}$, i.e. that for any $p \in \mathcal{M}$ and any $X, Y \in T_p\mathcal{M}$ which are not collinear, $K(X,Y)|_p$ has the same value K. A Riemannian manifold with the last property is called a **space form**. Show that the Riemann curvature tensor satisfies in this case:

$$R(X,Y,Z,W) = K \cdot \big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big).$$

Remark. For $n \leqslant 3$, every isotropic Riemannian manifold is a space form; this is not true for $n \geqslant 4$.

Solution. (a) Using the fact that

$$\nabla^{(h)}_{F_*X}(F_*Y) = F_*\big(\nabla^{(g)}_X Y\big) \quad \text{for all } X,Y \in \Gamma(\mathcal{M})$$

(which was proved when $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is an isometry in the solution of Exercise 6.1), together with the relation $[F_*X, F_*Y] = F_*([X,Y])$(which is true for any smooth map F), we have:

$$\begin{aligned}
R_h(F_*X,\, F_*Y,\, F_*Z,\, F_*W) &\doteq \big\langle R_h(F_*X, F_*Y)(F_*W),\, F_*Z\big\rangle_h \\
&= \big\langle \nabla^{(h)}_{F_*X}\nabla^{(h)}_{F_*Y}(F_*W) - \nabla^{(h)}_{F_*Y}\nabla^{(h)}_{F_*X}(F_*W) - \nabla^{(h)}_{[F_*X,F_*Y]}(F_*W),\, F_*Z\big\rangle_h \\
&= \big\langle \nabla^{(h)}_{F_*X}\nabla^{(h)}_{F_*Y}(F_*W) - \nabla^{(h)}_{F_*Y}\nabla^{(h)}_{F_*X}(F_*W) - \nabla^{(h)}_{F_*([X,Y])}(F_*W),\, F_*Z\big\rangle_h \\
&= \big\langle \nabla^{(h)}_{F_*X}\big(F_*(\nabla^{(g)}_Y W)\big) - \nabla^{(h)}_{F_*Y}\big(F_*(\nabla^{(g)}_X W)\big) - F_*(\nabla^{(g)}_{[X,Y]}W),\, F_*Z\big\rangle_h \\
&= \big\langle F_*\big(\nabla^{(g)}_X\nabla^{(g)}_Y W\big) - F_*\big(\nabla^{(g)}_Y\nabla^{(g)}_X W\big) - F_*(\nabla^{(g)}_{[X,Y]}W),\, F_*Z\big\rangle_h \\
&= \big\langle F_*\big(\nabla^{(g)}_X\nabla^{(g)}_Y W - \nabla^{(g)}_Y\nabla^{(g)}_X W - \nabla^{(g)}_{[X,Y]}W\big),\, F_*Z\big\rangle_h \\
&= \big\langle \nabla^{(g)}_X\nabla^{(g)}_Y W - \nabla^{(g)}_Y\nabla^{(g)}_X W - \nabla^{(g)}_{[X,Y]}W,\, Z\big\rangle_g \\
&= \big\langle R_g(X,Y)W,\, Z\big\rangle_h = R_g(X,Y,Z,W),
\end{aligned}$$

where, in passing to the second to last line above, we used the fact that F is an isometry.

(b) Let $p, p' \in \mathcal{M}$ and $X, Y \in T_p\mathcal{M}$, $X', Y' \in T_{p'}\mathcal{M}$, such that X is not collinear with Y and similarly for X' and Y'. By our assumption, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ with $F(p) = p'$ such that $F_*X, F_*Y \in \operatorname{span}\{X', Y'\}$. Since F_* does not have a kernel and X, Y are not collinear, F_*X, F_*Y cannot be collinear; thus, there exist $a, b, c, d \in \mathbb{R}$ with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ such that

$$F_*X = aX' + bY', \quad F_*Y = cX' + dY'.$$

Note that, as we calculated in class, the symmetries of the Riemann curvature tensor imply that, in this case,

$$R(F_*X, F_*Y, F_*X, F_*Y) = \left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^2 R(X', Y', X', Y')$$

and we can similarly calculate:

$$\|F_*X\|^2\|F_*Y\|^2 - \langle F_*X, F_*Y\rangle^2 = \left(\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^2 \big(\|X'\|^2\|Y'\|^2 - \langle X', Y'\rangle^2\big).$$

Therefore, using the definition of the sectional curvature and part (a) of this exercise (since F is an isometry), we calculate that

$$\begin{aligned} K_p(X,Y) &\doteq \frac{R(X,Y,X,Y)}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2} \\ &= \frac{R(F_*X,F_*Y,F_*X,F_*Y)}{\|F_*X\|^2\|F_*Y\|^2 - \langle F_*X,F_*Y\rangle^2} \\ &= \frac{R(X',Y',X',Y')}{\|X'\|^2\|Y'\|^2 - \langle X',Y'\rangle^2} \\ &\doteq K_{p'}(X',Y'). \end{aligned}$$

Thus, we proved that $(\mathcal{M}, g)$ has constant sectional curvature, i.e. there exists some $K \in \mathbb{R}$ such that

$$K_p(X,Y) = K \quad \text{for all } p \in \mathcal{M} \text{ and non-collinear } X,Y \in T_p\mathcal{M}.$$

This implies that, for any $X, Y \in \Gamma(\mathcal{M})$:

$$R(X,Y,X,Y) = K \cdot \big(g(X,X)g(Y,Y) - (g(X,Y))^2\big). \tag{1}$$

We will now show that, using the symmetries of the Riemann curvature tensor, we can express $R(X,Y,Z,W)$ as a linear combination of terms of the form $R(U,V,U,V)$ for suitable vector fields U,V defined in terms of X,Y,Z,W; these terms satisfy the relation (1). Recall that if b is a symmetric bilinear form, then $b(X,Y)$ can be expressed as a linear combination of terms of the form $b(V,V)$ via the well-known polarizing identity

$$b(X,Y) = \frac{1}{4}\Big(b(X+Y,X+Y) - b(X-Y,X-Y)\Big).$$

Replacing Y with tY, dividing the whole expression by t and then taking the limit $t \to 0$, we obtain the alternative (but less standard) differential form of the polarizing identity:

$$b(X,Y) = \lim_{t\to 0}\frac{1}{4t}\Big(b(X+tY,X+tY) - b(X-tY,X-tY)\Big) = \frac{1}{2}\frac{\partial}{\partial t}b(X+tY,X+tY)\Big|_{t=0}.$$

We can obtain a similar identity for $(0,4)$-tensors with the symmetries of the Riemann curvature tensor: Let us start by computing the following expression (using the symmetries of R in the last equality below):

$$\begin{aligned} &\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big(R(V_1+tV_3,V_2+sV_4,V_1+tV_3,V_2+sV_4)\Big)\Big|_{t,s=0} \qquad (2)\\ &\qquad = R(V_4,V_3,V_1,V_2) + R(V_3,V_2,V_1,V_4) + R(V_1,V_4,V_3,V_2) + R(V_1,V_2,V_3,V_4) \\ &\qquad = 2R(V_1,V_2,V_3,V_4) - 2R(V_1,V_4,V_2,V_3) \end{aligned}$$

(in order to verify the above identity, note that if $f(t,s)$ is a polynomial in t,s, then $\partial_t\partial sf|_{t,s=0}$ gives the coefficient of the term $t\cdot s$). Therefore, combining the above identity for $(V_1,V_2,V_3,V_4) = (X,Y,Z,W)$ and $(V_1,V_2,V_3,V_4) = (Y,X,Z,W)$, we obtain:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big(R(X+tZ,Y+sW,X+tZ,Y+sW) - R(Y+tZ,X+sW,Y+tZ,X+sW)\Big)\Big|_{t,s=0} \tag{3}$$

$$
\begin{aligned}
&= 2\Big(R(X,Y,Z,W) - R(X,W,Y,Z)\Big) - 2\Big(R(Y,X,Z,W) - R(Y,W,X,Z)\Big) \\
&= 6R(X,Y,Z,W), \qquad (4)
\end{aligned}
$$

where, in passing to the last line above, we used the fact that $R(X,Y,Z,W) = -R(Y,X,Z,W)$ and $R(Y,W,X,Z) = -R(X,Z,W,Y)$, together with the first Bianchi identity

$$R(X,Y,Z,W) + R(X,W,Y,Z) + R(X,Z,W,Y) = 0.$$

The relation (1) implies that

$$
\begin{aligned}
R(X+tZ,&Y+sW, X+tZ, Y+sW) \\
&= K\cdot\Big(g(X+tZ,X+tZ)g(Y+sW,Y+sW) - \big(g(X+tZ,Y+sW)\big)^2\Big) \\
&= K\cdot\Big(\big(g(X,X)+2tg(X,Z)+t^2g(Z,Z)\big)\big(g(Y,Y)+2sg(Y,W)+s^2g(W,W)\big) \\
&\qquad - \big(g(X,Y)+tg(Y,Z)+sg(X,W)+stg(Z,W)\big)^2\Big).
\end{aligned}
$$

Thus, substituting the above expression for the curvature terms in the left hand side of the polarizing identity (3), we calculate that

$$R(X,Y,Z,W) = K\cdot\big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\big).$$

10.3 (a) Compute the sectional curvature of the hyperbolic plane $(\mathbb{H}^2, g_{\mathbb{H}})$. *Hint: Use the expression of $g_{\mathbb{H}}$ in polar coordinates.)*

(b) Compute the Riemann curvature tensor, Ricci tensor and sectional curvature tensor of $(\mathbb{S}^n, g_{\mathbb{S}^n})$. *Hint: You can do the computations directly in one of the coordinate expressions of $g_{\mathbb{S}^n}$ that we've seen in the exercises, or note that $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is a space form.*

Solution. (a) In a previous exercise4, we calculated that, in polar coordinates around each point, $g_{\mathbb{H}}$ takes the form

$$g_{\mathbb{H}} = dr^2 + \sinh^2 r d\theta^2.$$

Thus, using Exercise 10.1 with $f(r,\theta) = \sinh(r)$, we calculate that

$$K = -\frac{1}{f}\partial_r^2 f = -1.$$

(b) The group $SO(n+1)$ of linear isometries of $(\mathbb{R}^{n+1}, g_E)$ also acts isometrically on $(\mathbb{S}^n, g_{\mathbb{S}^n)}) \subset (\mathbb{R}^{n+1}, g_E)$. It is easy to verify that for any $p, p' \in \mathbb{S}^n$ and any tangent 2-planes $\Pi \subset T_p\mathbb{S}^n \subset T_p\mathbb{R}^{n+1}$, $\Pi' \subset T_{p'}\mathbb{S}^n \subset T_{p'}\mathbb{R}^{n+1}$, there exists an $F \in SO(n+1)$ with $F(p) = p'$ and $F_*(\Pi) = \Pi^*$ (one way to see this is by noting that, for any $p \in \mathbb{S}^n$ and 2-planes $\Pi \subset T_p\mathbb{S}^n \subset T_p\mathbb{R}^{n+1}$, the plane Π is orthogonal to the unit vector e_0 connecting 0 to p; therefore, if we extend e_0 to an orthonormal base $\{e_\alpha\}_{\alpha=0}^n$ such that e_1, e_2 are parallel to the 2-plane Π, and we define a similar basis $\{e'_\alpha\}_{\alpha=0}^n$ associated to p'

and Π'). Therefore, Exercise 10.2 implies that $(\mathbb{S}^n, g_{\mathbb{S}^n)})$ is a space form and, hence the Riemann curvature tensor of $g_{\mathbb{S}^n}$ takes the form

$$R(X,Y,Z,W) = K \cdot \big(g_{\mathbb{S}^n}(X,Z)g_{\mathbb{S}^n}(Y,W) - g_{\mathbb{S}^n}(X,W)g_{\mathbb{S}^n}(Y,Z)\big) \tag{5}$$

for some constant K on $\mathbb{S}^n$. There are two ways to compute K (and show it is equal to $+1$):

- In the local coordinates $(x^1, \dots, x^n)$ coming from the stereographic projection (see Exercise 2.3), the metric $g_{\mathbb{S}^n}$ takes the form

  $$g_{\mathbb{S}^n} = \frac{4}{(1+|x|^2)^2}\sum_{i=1}^n (dx^i)^2$$

  (where $|x|^2 = \sum_{i=1}^n (x^i)^2$). Therefore, at the point p with coordinates $(x^1, \dots, x^n) = (0, \dots, 0)$, we compute that $(g_{\mathbb{S}^n})_{ij}|_p = 4\delta_{ij}$, $\partial_k (g_{\mathbb{S}^n})_{ij}|_p = 0$, $\partial_k\partial_l (g_{\mathbb{S}^n})_{ij}|_p = -16\delta_{ij}\delta_{kl}$ and, therefore,

  $$R_{1212}|_p = 16.$$

  Therefore, evaluating (5) at p for $X = Z = \partial_1$ and $Y = W = \partial_2$, we compute that

  $$K = \frac{R_{1212}|_p}{(g_{\mathbb{S}^n})_{11}|_p (g_{\mathbb{S}^n})_{22}|_p - ((g_{\mathbb{S}^n})_{12}|_p)^2} = 1.$$

- For $n = 2$, we have calculated in Exercise 6.4 that, in polar coordinates around each point, $g_{\mathbb{S}^2}$ takes the form

  $$g_{\mathbb{S}^2} = dr^2 + \sin^2 r d\theta^2.$$

  Thus, using Exercise 10.1 with $f(r,\theta) = \sin(r)$, we calculate that, in this case,

  $$K = -\frac{1}{f}\partial_r^2 f = +1.$$

  In dimensions $n > 2$, using the fact that the intersection of any 3-dimensional vector space V of $\mathbb{R}^{n+1}$ with $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a totally geodesic 2-sphere S_V of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ which is isometric to $(\mathbb{S}^2, g_{\mathbb{S}^2})$; thus, as a corollary of the Gauss equation (that we will see next week), the sectional curvature of $(\mathbb{S}^n, g_{\mathbb{S}^n})$ with respect to any tangent 2-plane to S_V is the same as the sectional curvature of the induced metric on S_V, and hence equal to $+1$.

10.4 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Consider the submanifold $\mathcal{M}_f$ of $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ which is the graph of f, i.e.

$$\mathcal{M}_f = \big\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : \, t = f(x)\big\}.$$

Compute the second fundamental form and the Riemann curvature tensor of the induced metric on $\mathcal{M}_f$. (*Hint: You might want to use the Gauss equation for the latter calculation.*)

Solution. Let us fix a Cartesian coordinate system $(x^1,\dots,x^n)$ on $\mathbb{R}^n$; we will use the notation $(t;\bar{x}) \doteq (t,\bar{x}^1,\dots,\bar{x}^n)$ for the corresponding Cartesian coordinates on $\mathbb{R}^{n+1}=\mathbb{R}\times\mathbb{R}^n$. We will work in the parametrization of $\mathcal{M}_f$ by $\mathbb{R}^n$ obtained by viewing $\mathcal{M}_f$ as the graph of the function f: We will define the coordinate chart $\Phi:\mathcal{M}_f\to\mathbb{R}^n$ to be the projection $\Phi\big((f(\bar{x});\bar{x})\big)=x$; the corresponding parametrization map $\Phi^{-1}:\mathbb{R}^n\to\mathcal{M}_f\subset\mathbb{R}^{n+1}$ is given by $\Phi^{-1}(x)=(f(x);x)$.

Remark. Note that, via the coordinate chart Φ, functions on $\mathcal{M}_f=\{t-f(\bar{x})=0\}\subset\mathbb{R}^{n+1}$ can also be viewed as functions of the variables $(x^1,\dots,x^n)$, and vice versa; we will identify a function h in the former class with its coordinate expression $h\circ\Phi^{-1}$.

For any $p\in\mathcal{M}_f$, the tangent space $T_p\mathcal{M}_f\subset T_p\mathbb{R}^{n+1}$ is spanned by the coordinate vector fields $X_i=(\Phi^{-1})^*\frac{\partial}{\partial x^i}$, $i=1,\dots,n$. The vectors X_i have the following expression with respect to the Cartesian frame $\frac{\partial}{\partial t},\frac{\partial}{\partial\bar{x}^1},\dots,\frac{\partial}{\partial\bar{x}^n}$ of $T_p\mathbb{R}^{n+1}$:

$$X_i=\partial_i f\frac{\partial}{\partial t}+\frac{\partial}{\partial\bar{x}^i}. \tag{6}$$

(you can verify that $X_i(t-f(\bar{x}))=0$ and, hence, X_i is tangent to the submanifold $\mathcal{M}_f=\big\{t-f(\bar{x})=0\big\}$).

Remark. If h is a function on $\mathcal{M}_f$ defined in terms of the Cartesian variables $(t;\bar{x})$ of $\mathbb{R}^{n+1}$ and $h\circ\Phi^{-1}$ is its coordinate expression, then we trivially have that

$$X_i(h)=\partial_i(h\circ\Phi^{-1}).$$

Since, as stated in the previous remark, we will identity h with $h\circ\Phi^{-1}$, we will use the notation $\partial_i h$ and $X_i h$ interchangeably for functions on $\mathcal{M}_f$ to denote either $X_i(h)$ or $\partial_i(h\circ\Phi^{-1})$. This is a standard notational convention when dealing with coordinate expressions on submanifolds.

For any $p\in\mathcal{M}_f$, the normal space $(T_p\mathcal{M}_f)^\perp\subset T_p\mathbb{R}^{n+1}$ is spanned by the unit vector $\hat{n}$ which is perpendicular to X_i for $i=1,\dots,n$. Hence, we can compute the Cartesian components of $\hat{n}=\hat{n}^t\frac{\partial}{\partial t}+\hat{n}^i\frac{\partial}{\partial\bar{x}^i}$ via the relations

$$\begin{cases}\langle\hat{n},\hat{n}\rangle_{\mathbb{R}^{n+1}}=1,\\ \langle\hat{n},X_i\rangle_{\mathbb{R}^{n+1}}=0\quad\text{for } i=1,\dots,n.\end{cases}$$

The above system of equations has two solutions (which can be found by first solving $\langle\hat{n},X_i\rangle_{\mathbb{R}^{n+1}}=0$ to express each of the $\hat{n}^j$'s in terms of $\hat{n}^t$ and then using the quadratic equation $\langle\hat{n},\hat{n}\rangle_{\mathbb{R}^{n+1}}=1$ to find a value for $\hat{n}^t$):

$$\hat{n}=\pm\Big(\frac{1}{\sqrt{1+|df|^2}}\frac{\partial}{\partial t}-\sum_{j=1}^n\frac{\partial_j f}{\sqrt{1+|df|^2}}\frac{\partial}{\partial\bar{x}^j}\Big) \tag{7}$$

where $|df|^2=\sum_{i=1}^n(\partial_i f)^2$. From now one, we will fix a coorientation for $\mathcal{M}_f$ by choosing $\hat{n}$ to be equal to (7) with the $+$ sign, i.e. $\hat{n}^t=\frac{1}{\sqrt{1+|df|^2}}$ and $\hat{n}^j=\frac{\partial_j f}{\sqrt{1+|df|^2}}$, $j=1,\dots,n$.

The first fundamental form of $\mathcal{M}_f$ (i.e. the induced metric on $\mathcal{M}_f\subset(\mathbb{R}^{n+1},g_E)$) takes the form:

$$\bar{g}_{ij}\doteq g_E(X_i,X_j)=\Big\langle\partial_i f\frac{\partial}{\partial t}+\frac{\partial}{\partial\bar{x}^i},\,\partial_j f\frac{\partial}{\partial t}+\frac{\partial}{\partial\bar{x}^j}\Big\rangle_{\mathbb{R}^{n+1}}=\partial_i f\partial_j f+\delta_{ij}.$$

Note that, denoting with ∇ the Levi-Civita connection of g_E (i.e. the flat connection on $\mathbb{R}^{n+1}$), we have

$$\begin{aligned}\nabla_{X_i}\hat{n} &= X_i(\hat{n}^t)\frac{\partial}{\partial t} + X_i(\hat{n}^j)\frac{\partial}{\partial \bar{x}^j}\\ &= \partial_i(\hat{n}^t)\frac{\partial}{\partial t} + \partial_i(\hat{n}^j)\frac{\partial}{\partial \bar{x}^j}\\ &= \partial_i\Big(\frac{1}{\sqrt{1+|df|^2}}\Big)\frac{\partial}{\partial t} - \sum_{j=1}^{n}\partial_i\Big(\frac{\partial_j f}{\sqrt{1+|df|^2}}\Big)\frac{\partial}{\partial \bar{x}^j}\end{aligned}$$

Thus, the scalar second fundamental form b can be computed by the formula

$$\begin{aligned}b_{ij} \doteq b(X_i,X_j) &= -g_E(\nabla_{X_i}\hat{n}, X_j)\\ &= \Big\langle \partial_i\Big(\frac{1}{\sqrt{1+|df|^2}}\Big)\frac{\partial}{\partial t} - \sum_{k=1}^{n}\partial_i\Big(\frac{\partial_k f}{\sqrt{1+|df|^2}}\Big)\frac{\partial}{\partial \bar{x}^k},\, \partial_j f\frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{x}^j}\Big\rangle_{\mathbb{R}^{n+1}}\\ &= \partial_i\Big(\frac{1}{\sqrt{1+|df|^2}}\Big)\partial_j f - \partial_i\Big(\frac{\partial_j f}{\sqrt{1+|df|^2}}\Big)\\ &= \frac{\partial_i\partial_j f}{\sqrt{1+|df|^2}}.\end{aligned}$$

The second fundamental form B then takes the form

$$B_{ij} \doteq B(X_i,X_j) = b(X_i,X_j)\hat{n} = \frac{\partial_i\partial_j f}{\sqrt{1+|df|^2}}\hat{n}.$$

Finally, using the Gauss equation and the fact that the Riemann curvature tensor R_E of g_E vanishes identically, we have

$$\bar{R}_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk} = \frac{\partial_i\partial_k f\cdot\partial_j\partial_l f - \partial_i\partial_l f\cdot\partial_j\partial_k f}{1+|df|^2}$$

(recall our convention that $\bar{R}_{ijkl} = \bar{R}(\partial_i,\partial_j,\partial_k,\partial_l) = -\bar{g}\big(R(\partial_i,\partial_j)\partial_k,\partial_l\big)$).

10.5 Let $(\mathcal{M}^n, g)$ be a smooth Riemannian manifold and let p be a point on $\mathcal{M}$. For any given $0<\bar{r}<\iota(p)$, let us consider the open neighborhood $\mathcal{U} = \exp_p\Big(\{v : \|v\| < \bar{r}\}\Big)$ of p. Recall that $\mathcal{U}\setminus\{p\}$ is prametrized by the polar coordinates $(r,\omega)\in(0,\bar{r})\times\mathbb{S}^{n-1}$, where $r(\cdot) = dist_g(\cdot)$. Recall also that, in any local coordinate chart $(x^1,\dots,x^{n-1})$ on $\mathbb{S}^{n-1}$, the metric g in the $(r,x^1,\dots,x^{n-1})$ coordinate system takes the form

$$g = dr^2 + r^2\bar{g}_{ij}[r]dx^i dx^j,$$

where $\bar{g}_{ij}[r] \xrightarrow{r\to 0} (g_{\mathbb{S}^{n-1}})_{ij}$ and $\partial_r\bar{g}_{ij}[r] \xrightarrow{r\to 0} 0$ (with $g_{\mathbb{S}^{n-1}}$ denoting the standard round metric on the unit sphere).

(a) Show that
$$\partial_r\big(r^2\bar{g}_{ij}[r]\big) = -2b_{ij}[r],$$
where $b[\rho]$ is the scalar second fundamental form of the hypersurface $S_\rho = \{r = \rho\}$ with respect to the coorientation determined by grad$r$. *(Hint: Use Exercise 11.1.b.)*

(b) Show that
$$\partial_r b_{ij}[r] + r^{-2}\bar{g}^{ab}[r]\cdot b_{ia}[r]\cdot b_{jb}[r] = R_{rirj},$$
where R is the Riemann curvature tensor of g.

* (c) Show that if $R \equiv 0$, then $\bar{g}_{ij}[r] = (g_{\mathbb{S}^{n-1}})_{ij}$ for all $r \in (0, \bar{r})$. Deduce, in this case, that g is isometric to the flat metric g_E. *(Hint: Show that, in this case, the tensor $M^i_j[r] = r^{-2}\bar{g}^{ia}[r]\cdot b_{jb}[r]$ on S_r satisfies, with respect to r, the matrix Ricatti ODE $\partial_r M - M^2 = 0$. What is $\lim_{r\to 0} M$?)*

Solution. (a) Since, in the $(r, x^1, \dots, x^{n-1})$ coordinate chart, the metric takes the form
$$g = dr^2 + r^2\bar{g}_{ij}dx^i dx^j,$$
(where, for any $i, j \in \{1, \dots, n-1\}$, $\bar{g}_{ij}$ is a function depending on $r, x^1, \dots, x^{n-1}$), we calculate that
$$g_{rr} = 1, \quad , q_{ij} = r^2\bar{g}_{ij}, \quad g_{ri} = 0$$
and
$$g^{rr} = 1, \quad g^{ij} = r^{-2}\bar{g}^{ij}, \quad g^{ri} = 0$$
(where $\bar{g}^{ij}$ are the components of the inverse matrix of $[\bar{g}_{ij}]$). Therefore, for the coordinate 1-form dr, we can readily compute that
$$\text{grad}r \doteq dr^\sharp = \frac{\partial}{\partial r}$$
(since $(dr^\sharp)^r = g^{rr}dr_r + g^{ri}dr_i = dr_r = 1$ and $(dr^\sharp)^i = g^{ir}dr_r + g^{ij}dr_j = 0$). Using Exercise 11.1.b, we can compute that the scalar second fundamental form of the level set $S_\rho = \{r = \rho\}$ of the function r takes the form
$$b(X,Y) = -\frac{Hess[r](X,Y)}{\|\text{grad}r\|} = -Hess[r](X,Y) \quad \text{for all } X, Y \in \Gamma(\mathcal{M}, S_\rho).$$
Note that the tangent space at any point of S_ρ is spanned by the coordinate vector fields $\frac{\partial}{\partial x^i}$, $i = 1, \dots, n-1$ (which are also the coordinate vector fields for the parametrization of S_ρ by $(x^1, \dots, x^{n-1})$). Therefore, applying the formula for the expression of the Hessian in local coordinates from Ex. 11.1.a, we calculate in the $(r, x^1, \dots, x^{n-1})$ coordinate system:
$$b_{ij} = -Hess[r](\partial_i, \partial_j) = -Hess[r]_{ij} = -\Big(\partial_i\partial_j r - \Gamma^\beta_{ij}\partial_\beta r\Big) = \Gamma^r_{ij}$$
(since $\partial_r r = 1$, $\partial_i r = 0$).

Remark. For the rest of this exercise, we will use the convention that Latin indices $i, j, k, l, \dots$, range over the coordinates $(x^1, \dots, x^{n-1})$, while Greek indices $\beta, \beta, \dots$ range over $(r, x^1, \dots, x^{n-1})$.

From our computation of the components of g and g^{-1}, we have:

$$\begin{aligned}\Gamma^r_{ij} &= \frac{1}{2} g^{r\beta}\big(\partial_i g_{\beta j} + \partial_j g_{\beta i} - \partial_\beta g_{ij}\big)\\ &= \frac{1}{2} g^{rr}\big(\partial_i g_{rj} + \partial_j g_{ri} - \partial_r g_{ij}\big) + \frac{1}{2} g^{rk}\big(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\big)\\ &= -\frac{1}{2}\partial_r g_{ij} + 0\\ &= -\frac{1}{2}\partial_r (r^2 \bar{g}_{ij}).\end{aligned}$$

Therefore, combining the above relations, we get the desired identity

$$b_{ij} = -\frac{1}{2}\partial_r (r^2 \bar{g}_{ij}).$$

(b) Using the expression for the components of the Riemann curvature tensor in local coordinates

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\lambda}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\delta\lambda}\Gamma^\lambda_{\beta\gamma},$$

together with our calculation of the components of g and g^{-1}, we can readily compute:

$$\begin{aligned}R_{rirj} &= g_{r\beta} R^\beta{}_{irj}\\ &= g_{rr} R^r{}_{irj} + g_{rk} R^k{}_{irj}\\ &= R^r{}_{irj}\\ &= \partial_r \Gamma^r_{ij} - \partial_j \Gamma^r_{ir} + \Gamma^r_{r\lambda}\Gamma^\lambda_{ij} - \Gamma^r_{j\lambda}\Gamma^\lambda_{ir}.\end{aligned}$$

We already computed in part (a) that

$$\Gamma^r_{ij} = -\frac{1}{2}\partial_r (r^2 \bar{g}_{ij}) = b_{ij}.$$

For the rest of the Christoffel symbols appearing in the above expression, we can similarly compute:

$$\Gamma^r_{rr} = \frac{1}{2} g^{r\beta}\big(\partial_r g_{\beta r} + \partial_r g_{\beta r} - \partial_\beta g_{rr}\big) = \frac{1}{2} g^{rr}\big(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\big) + 0 = 0,$$

$$\Gamma^r_{ri} = \frac{1}{2} g^{r\beta}\big(\partial_r g_{\beta i} + \partial_i g_{\beta r} - \partial_\beta g_{ir}\big) = \frac{1}{2} g^{rr}\big(\partial_r g_{ri} + \partial_i g_{rr} - \partial_r g_{ir}\big) + 0 = 0$$

and

$$\Gamma^k_{ir} = \frac{1}{2} g^{k\beta}\big(\partial_i g_{\beta r} + \partial_r g_{\beta i} - \partial_\beta g_{ir}\big) = \frac{1}{2} g^{kl}\big(\partial_r g_{li} + \partial_i g_{lr} - \partial_l g_{ir}\big) + 0 = \frac{1}{2} g^{kl}\partial_r g_{li} = r^{-2}\bar{g}^{kl}\partial_r (r^2 \bar{g}_{li}) = -r^{-2}\bar{g}^{kl} b_{li}.$$

Therefore,

$$R_{rirj} = \partial_r \Gamma^r_{ij} - \Gamma^r_{jk}\Gamma^k_{ir} + 0 = \partial_r b_{ij} + r^{-2} g^{kl} b_{ik} b_{jl}.$$

(c) In the case when the Riemann curvature tensorvanishes identically, the system of equations derived in parts (a) and (b) for $\bar{g}$ and b becomes:

$$\begin{cases} \partial_r\big(r^2\bar{g}_{ij}\big) = -2b_{ij}, \\ \partial_r b_{ij} + r^{-2}\bar{g}^{ab}\cdot b_{ia}\cdot b_{jb} = 0. \end{cases} \tag{8}$$

Let us consider the $(1,1)$ tensor M on S_r obtained by raising one of the indices of b, namely

$$M^i_j = g^{ik}b_{kj} = r^{-2}\bar{g}^{ik}b_{kj}.$$

Thus, we can readily compute:

$$\begin{aligned} \partial_r M^i_j &= \partial_r(r^{-2}\bar{g}^{ik}b_{kj}) \\ &= \partial_r(r^{-2}\bar{g}^{ik})b_{kj} + r^{-2}\bar{g}^{ik}\partial_r b_{kj} \\ &= \partial_r(r^{-2}\bar{g}^{ik})b_{kj} - r^{-2}\bar{g}^{ik}r^{-2}\bar{g}^{ab}\cdot b_{ka}\cdot b_{jb} \\ &= \partial_r(r^{-2}\bar{g}^{ik})b_{kj} - (r^{-2}\bar{g}^{ik}\cdot b_{ka})\cdot(r^{-2}\bar{g}^{ab}b_{jb}) \qquad = \partial_r(r^{-2}\bar{g}^{ik})b_{kj} - M^i_a M^a_j, \end{aligned}$$

where, in the second to last step, we used the equation for $\partial_r b_{ij}$ above. Recall that, if A is a matrix-valued function depending on a parameter s, then

$$\frac{d}{ds}(A^{-1}) = -A^{-1}\frac{dA}{ds}A^{-1}.$$

Since $[r^{-2}\bar{g}^{ik}]$ are the components of the inverse matrix of $[r^2\bar{g}_{ik}]$, the above relation formula implies that

$$\begin{aligned} \partial_r\big(r^{-2}\bar{g}^{ik}\big) &= -\big(r^{-2}\bar{g}^{il}\big)\cdot\big(\partial_r(r^2\bar{g}_{lm})\big)\cdot\big(r^{-2}\bar{g}^{mk}\big) \\ &= 2\big(r^{-2}\bar{g}^{il}\big)\cdot b_{lm}\cdot\big(r^{-2}\bar{g}^{mk}\big) \\ &= 2M^i_m\cdot\big(r^{-2}\bar{g}^{mk}\big) \end{aligned}$$

where, in the second to last line above, we made use of (8) for $\partial_r(r^2\bar{g}_{lm})$. Substituting in the right hand side equation for $\partial_r M^i_j$, we therefore obtain:

$$\begin{aligned} \partial_r M^i_j &= 2M^i_m\cdot r^{-2}\bar{g}^{mk}\cdot b_{kj} - M^i_a M^a_j \\ &= 2M^i_m M^m_j - M^i_a M^a_j \\ &= M^i_m M^m_j, \end{aligned}$$

or, in matrix notation, for the matrix $M = [M^i_j]$:

$$\partial_r M = M^2.$$

Using the formula for the derivative of the inverse of a matrix, the above equation is equivalent to

$$\partial_r\big(M^{-1}\big) = -\mathbb{I}. \tag{9}$$

As $r \to 0$, we have that $\bar{g}_{ij} \to (g_{\mathbb{S}^{n-1}})_{ij}$ and $\partial_r \bar{g}_{ij} \to 0$. Therefore, since $b_{ij} = -\frac{1}{2}\partial_r(r^2\bar{g}_{ij})$, we have that

$$\begin{aligned}
lim_{r\to 0}\Big(rM^i_j\Big) &= lim_{r\to 0}\Big(r^{-1}\bar{g}^{ik}b_{kj}\Big)\\
&= -\frac{1}{2}lim_{r\to 0}\Big(r^{-1}\bar{g}^{ik}\partial_r(r^2\bar{g}_{kj})\Big)\\
&= -\frac{1}{2}lim_{r\to 0}\Big(r^{-1}\bar{g}^{ik}\big(2r\bar{g}_{kj} + r^2\partial_r\bar{g}_{kj}\big)\Big)\\
&= -lim_{r\to 0}\Big(\bar{g}^{ik}\bar{g}_{kj}\Big) - \frac{1}{2}lim_{r\to 0}\Big(r\partial_r\bar{g}_{kj}\Big)\\
&= -\delta^i_j,
\end{aligned}$$

or, in matrix notation,

$$\lim_{r\to 0}(rM) = -\mathbb{I}$$

and, hence,

$$\lim_{r\to 0}(M^{-1}) = \lim_{r\to 0}(r(rM)^{-1}) = 0.$$

Integrating the ODE (9) from $r = 0$ using the above initial condition, we infer that

$$M^{-1} = -r\mathbb{I} \quad \Leftrightarrow \quad M = r^{-1}\mathbb{I}.$$

In view of our definition $M^i_j = r^{-2}\bar{g}^{ik}b_{kj} = -\frac{1}{2}r^{-2}\bar{g}^{ik}\partial_r\big(r^2\bar{g}_{jk}\big)$, the above is equivalent to the statement that

$$-\frac{1}{2}r^{-2}\bar{g}^{ik}\partial_r\big(r^2\bar{g}_{jk}\big) = -r^{-1}\delta^i_j.$$

After expanding $\partial_r\big(r^2\bar{g}_{jk}\big) = 2r\bar{g}_{jk} + r^2\partial_r\bar{g}_{jk}$ and multiplying both sides with $\bar{g}_{il}$ (and summing over i), we obtain:

$$-\frac{1}{2}r^{-2}\big(2r\bar{g}_{jl} + r^2\partial_r\bar{g}_{jl}\big) = -r^{-1}\bar{g}_{jl} \quad \Leftrightarrow \quad \partial_r\bar{g}_{jl} = 0.$$

Since $\lim_{r\to 0}\bar{g}_{ij} = (g_{\mathbb{S}^{n-1}})_{ij}$, we deduce that

$$\bar{g}_{ij} = (g_{\mathbb{S}^{n-1}})_{ij}.$$

Therefore,

$$g = dr^2 + r^2(g_{\mathbb{S}^{n-1}})_{ij}dx^i dx^j = g_E.$$

10.6 Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Riemannian manifolds and let $(\mathcal{M}, g) = (\mathcal{M}_1\times\mathcal{M}_2, g_1\oplus g_2$ be their Riemannian product; the metric $g_1\oplus g_2$ is defined so that, for any $p = (p_1, p_2) \in \mathcal{M}_1\times\mathcal{M}_2$ and any $X, Y \in T_p\mathcal{M} \simeq T_{p_1}\mathcal{M}_1\oplus T_{p_2}\mathcal{M}_2$, if $X = X_1+X_2$ and $Y = Y_1+Y_2$ is their corresponding decomposition into tangent vectors tangential to $\mathcal{M}_1\times\{p_2\}$ and $\{p_1\}\times\mathcal{M}_2$ then

$$g(X, Y) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

(in other words, $\mathcal{M}_1\times\{p_2\}$ and $\{p_1\}\times\mathcal{M}_2$ intersect orthogonally and $\mathcal{M}_1 \to \mathcal{M}_1\times\{p_2\}$ and $\mathcal{M}_2 \to \{p_1\}\times\mathcal{M}_2$ are isometric embeddings).

(a) Compute the Riemann curvature tensor R of $(\mathcal{M}, g)$ in terms of the Riemann curvature tensors R_i of $(\mathcal{M}_i, g_i)$, $i = 1, 2$.

(b) Show that the sectional curvature of $(\mathcal{M}, g)$ cannot be strictly positive or strictly negative for all tangent 2-planes.

(*c) Show that there exists a surface in $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^2})$ which is totally geodesic (i.e. has vanishing second fundamental form) and is isometric to the flat torus $(\mathbb{T}^2, g_E)$.

Solution. (a) Let $p = (p_1, p_2) \in \mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$; note that any pair of curves $t \to \gamma_i(t) \in \mathcal{M}_i$ with $\gamma_i(0) = p_i$, $i = 1, 2$, can be identified with the curve $t \to \gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathcal{M}$, $\gamma(0) = p$. Through this identification, we can also identify $\dot\gamma(0)$ with $(\dot\gamma_1(0), \dot\gamma_2(0)$; thus, we naturally have $T_p\mathcal{M} \simeq T_{p_1}\mathcal{M}_1 \oplus T_{p_2}\mathcal{M}_2$, with $T_{p_1}\mathcal{M}_1 \subset T_p\mathcal{M}$ corresponding to the set of tangent directions at $t = 0$ of curves of the form $t \to (\gamma_1(t), p_2)$, $\gamma_1(0) = p_1$ (i.e. $T_{p_1}\mathcal{M}_1$ corresponds to the tangent space of the submanifold $\mathcal{M}_1 \times \{p_2\}$ at (p_1, p_2)), and similarly for $T_{p_2}\mathcal{M}_2$. For any vector field $V \in \Gamma(\mathcal{M})$, we will denote with $V = V_1 + V_2$ its decomposition into components tangential to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. If $(x^1, \dots, x^n)$ is a local coordinate chart on $\mathcal{U}_1 \subset \mathcal{M}_1$ and $(y^1, \dots, y^m)$ is a local coordinate chart on $\mathcal{U}_2 \subset \mathcal{M}_2$, then in the product coordinate chart $(x^1, \dots, x^n; y^1, \dots, y^m)$ on $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \subset \mathcal{M}$, the decomposition $V = V_1 + V_2$ corresponds to

$$V_1 = V^i \frac{\partial}{\partial x^i}, \quad V_2 = V^\alpha \frac{\partial}{\partial y^\alpha},$$

where we are using Latin letters $i, j, k, \dots$ to denote indices associated to the chart $(x^1, \dots, x^n)$ on $\mathcal{M}_1$ and Greek letters $\alpha, \beta, \gamma, \dots$ for indices associated to the chart $(y^1, \dots, y^m)$ on $\mathcal{M}_2$ *Note that, in general, both the components of V_1 and V_2 depend on both x^i and y^α.*

For any point $p \in \mathcal{M}$ and any $X, Y, Z, W \in T_p\mathcal{M}$, we are asked to compute $R(X, Y, Z, W)$ in terms of $R^{(1)}$, $R^{(2)}$ and the decompositions X_i, Y_i, Z_i, W_i, $i = 1, 2$, of X, Y, Z, W. In fact, we will show that

$$R(X, Y, Z, W) = R^{(1)}(X_1, Y_1, Z_1, W_1) + R^{(2)}(X_2, Y_2, Z_2, W_2). \tag{10}$$

To this end, let us fix a product coordinate chart $(x^1, \dots, x^n; y^1, \dots, y^m)$ on a neighborhood of p as above (recall our convention that Latin indices are associated with $(x^1, \dots, x^n)$, while Greek indices are associated with $(y^1, \dots, y^n)$. It is easy to verify (in view of the multilinearity of $R(\cdot, \cdot, \cdot, \cdot)$) that (10) will follow once we show that

$$R_{ijkl} = R^{(1)}_{ijkl}, \quad R_{\alpha\beta\gamma\delta} = R^{(2)}_{\alpha\beta\gamma\delta}$$

and that all the "mixed" components vanish, i.e.

$$R_{\alpha ijk} = R_{\alpha\beta ij} = R_{\alpha i\beta j} = R_{\alpha\beta\gamma i} = 0$$

(using the symmetries of R, the above implies that any component of R with mixed Greek and Latin indices vanishes).

Let us denote with $\nabla^{(1)}, \nabla^{(2)}$ the Levi-Civita connections of the Riemannian manifolds $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$. We will now express the Christoffel symbols of the Levi-Civita connection ∇ of $(\mathcal{M}, g)$

in terms of those of $\nabla^{(1)}, \nabla^{(2)}$. Note that our assumption that $g = g_1 \oplus g_2$ is equivalent to the statement that

$$g_{ij} = (g_1)_{ij}, \quad g_{\alpha\beta} = (g_2)_{\alpha\beta}, \quad g_{i\alpha} = 0$$

and, thus,

$$g^{ij} = (g_1)^{ij}, \quad g^{\alpha\beta} = (g_2)^{\alpha\beta}, \quad g^{i\alpha} = 0$$

and that the components $(g_1)_{ij}$ of g_1 depend only on $(x^1, \dots, x^n)$ (and similarly for g_2 and $(y^1, \dots, y^m)$).

1. The Christoffel symbols of the form Γ^i_{jk} or $\Gamma^\alpha_{\beta\gamma}$ (i.e. with indices lying entirely in one of the charts $(x^1, \dots, x^n)$ or $(y^1, \dots, y^m)$) can be computed as follows:

$$\begin{aligned}\Gamma^i_{jk} &\doteq \frac{1}{2} g^{il}\big(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}\big) + \frac{1}{2} g^{i\alpha}\big(\partial_j g_{\alpha k} + \partial_k g_{\alpha j} - \partial_\alpha g_{jk}\big)\\ &= \frac{1}{2}(g_1)^{il}\big(\partial_j (g_1)_{lk} + \partial_k (g_1)_{lj} - \partial_l (g_1)_{jk}\big) + 0\\ &= (\Gamma^{(1)})^i_{jk}\end{aligned}$$

   and, similarly,

$$\Gamma^\alpha_{\beta\gamma} = (\Gamma^{(2)})^\alpha_{\beta\gamma}.$$

2. For the Christoffel symbols of mixed type Γ^α_{ij}, we calculate

$$\begin{aligned}\Gamma^\alpha_{ij} &\doteq \frac{1}{2} g^{\alpha k}\big(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\big) + \frac{1}{2} g^{\alpha\beta}\big(\partial_i g_{\beta j} + \partial_j g_{\beta i} - \partial_\beta g_{ij}\big)\\ &= 0 + 0\end{aligned}$$

   (where we used the fact that $g^{\alpha k} = g_{\beta j} = g_{\beta i} = 0$ and that the components $(g_1)_{ij}$ of g_1 depend only on $(x^1, \dots, x^n)$). Similarly,

$$\Gamma^i_{\alpha\beta} = 0.$$

3. For the Christoffel symbols of mixed type $\Gamma^i_{\alpha j}$, we similarly have

$$\begin{aligned}\Gamma^i_{\alpha j} &\doteq \frac{1}{2} g^{ik}\big(\partial_\alpha g_{kj} + \partial_j g_{k\alpha} - \partial_k g_{\alpha j}\big) + \frac{1}{2} g^{i\beta}\big(\partial_\alpha g_{\beta j} + \partial_j g_{\alpha\beta} - \partial_\beta g_{\alpha j}\big)\\ &= 0 + 0\end{aligned}$$

   and, similarly,

$$\Gamma^\alpha_{\beta i} = 0.$$

Collecting the above calculations, we deduce that

$$\nabla_{\partial_i}\partial_j = \nabla^{(1)}_{\partial_i}\partial_j, \quad \nabla_{\partial_\alpha}\partial_\beta = \nabla^{(2)}_{\partial_\alpha}\partial_\beta, \quad \nabla_{\partial_i}\partial_\alpha = \nabla_{\partial_\alpha}\partial_i = 0. \tag{11}$$

Moreover, if $V \in \Gamma(\mathcal{M})$ is a vector field which is "decomposable", in the sence that, in the product coordinates, $V(x; y) = V_1(x) + V_2(y)$ (namely the components V^i are indpendent of y^α and V^α are indpendent of x^i), then, for any $X \in \Gamma(\mathcal{M})$:

$$(\nabla_X V)^i = X^j \partial_j V^i + X^\alpha \partial_\alpha V^i + \Gamma^i_{jk} X^j V^k + \Gamma^i_{\alpha k} X^\alpha V^k + \Gamma^i_{j\beta} X^j V^\beta + \Gamma^i_{\alpha\beta} X^\alpha V^\beta$$

$$= X^j \partial_j V^i + 0 + (\Gamma^{(1)})^i_{jk} X^j V^k + 0$$
$$= (\nabla^{(1)}_{X_1} V_1)^i$$

and, similarly,

$$(\nabla_X V)^\alpha = (\nabla^{(2)}_{X_2} V_2)^\alpha,$$

so that

$$\nabla_X V = \nabla^{(1)}_{X_1} V_1 + \nabla^{(2)}_{X_2} V_2. \tag{12}$$

Remark. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a curve in $\mathcal{M}$ with $\dot\gamma_1, \dot\gamma_2 \neq 0$. It is easy to verify that, in a product coordinate system in a small enough neighborhood $\mathcal{U}$ of a point p of γ, the tangent vector field $\dot\gamma$ can be extended (in a non-unique way) to a vector field on $\mathcal{U}$ which is decomposable in the above sense. Therefore, the acceleration of the curve γ around a point where $\dot\gamma_1, \dot\gamma_2 \neq 0$ satisfies

$$\nabla_{\dot\gamma}\dot\gamma = \nabla^{(1)}_{\dot\gamma_1}\dot\gamma_1 + \nabla^{(2)}_{\dot\gamma_2}\dot\gamma_2.$$

In particular, γ is a geodesic of $(\mathcal{M}, g)$ if and only if γ_1 and γ_2 are geodesics of $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$, respectively.

Note that the coordinate vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^\alpha}$ are decomposable (since their components are constant functions in (x, y), equal to 0 or 1), and the same is true for the vector fields $\nabla_{\partial_i}\partial_j$, $\nabla_{\partial_\alpha}\partial_\beta$ and $\nabla_{\partial_\alpha}\partial_i$, since, in view of (11), we have

$$(\nabla_{\partial_i}\partial_j)_1 = \nabla^{(1)}_{\partial_i}\partial_j = (\Gamma^{(1)})^k_{ij}\partial_k, \quad (\nabla_{\partial_i}\partial_j)_2 = 0$$
$$(\nabla_{\partial_\alpha}\partial_\beta)_1 = 0, \quad (\nabla_{\partial_\alpha}\partial_\beta)_2 = \nabla^{(2)}_{\partial_\alpha}\partial_\beta = (\Gamma^{(2)})^\gamma_{\alpha\beta}\partial_\gamma,$$
$$\nabla_{\partial_\alpha}\partial_i = \nabla_{\partial_i}\partial_\alpha = 0$$

and $(\Gamma^{(1)})^k_{ij}$ is a function of only $(x^1, \ldots, x^n)$, while $(\Gamma^{(2)})^\gamma_{\alpha\beta}$ is a function of only $(y^1, \ldots, y^m)$. Therefore, using the formula (12) for those vector fields, we have:

$$\begin{aligned}
R_{ijkl} &\doteq R(\partial_i, \partial_j, \partial_k, \partial_l) \\
&= -g\Big(\nabla_{\partial_i}(\nabla_{\partial_j}\partial_k) - \nabla_{\partial_j}(\nabla_{\partial_i}\partial_k), \partial_l\Big) \\
&= -g\Big(\big[\nabla^{(1)}_{(\partial_i)_1}((\nabla_{\partial_j}\partial_k)_1) + \nabla^{(2)}_{(\partial_i)_2}((\nabla_{\partial_j}\partial_k)_2)\big] - \big[\nabla^{(1)}_{(\partial_j)_2}((\nabla_{\partial_i}\partial_k)_1) - \big(\nabla^{(2)}_{(\partial_j)_2}((\nabla_{\partial_i}\partial_k)_2)\big], \partial_l\Big) \\
&= -g\Big(\nabla^{(1)}_{\partial_i}((\nabla_{\partial_j}\partial_k)_1) + 0 - \big(\nabla^{(1)}_{(\partial_j)_2}((\nabla_{\partial_i}\partial_k)_1) - 0, \partial_l\Big) \\
&= -g\Big(\nabla^{(1)}_{\partial_i}\nabla^{(1)}_{\partial_j}\partial_k - \nabla^{(1)}_{\partial_j}\nabla^{(1)}_{\partial_i}\partial_k, \partial_l\Big) \\
&= -g_1\Big(\big(\nabla^{(1)}_{\partial_i}\nabla^{(1)}_{\partial_j}\partial_k - \nabla^{(1)}_{\partial_j}\nabla^{(1)}_{\partial_i}\partial_k\big)_1, (\partial_l)_1\Big) - g_2\Big(\big(\nabla^{(1)}_{\partial_i}\nabla^{(1)}_{\partial_j}\partial_k - \nabla^{(1)}_{\partial_j}\nabla^{(1)}_{\partial_i}\partial_k\big)_2, (\partial_l)_2\Big) \\
&= -g_1\Big(\nabla^{(1)}_{\partial_i}\nabla^{(1)}_{\partial_j}\partial_k - \nabla^{(1)}_{\partial_j}\nabla^{(1)}_{\partial_i}\partial_k, \partial_l\Big) - 0 \\
&= R^{(1)}_{ijkl}
\end{aligned}$$

and, similarly, with the roles of $\mathcal{M}_1$ and $\mathcal{M}_2$ inverted:

$$R_{\alpha\beta\gamma\delta} = R^{(2)}_{\alpha\beta\gamma\delta}.$$

Moreover,

$$\begin{aligned}
R_{\alpha ijk} &\doteq R(\partial_\alpha, \partial_i, \partial_j, \partial_k) \\
&= -g\Big(\nabla_{\partial_\alpha}(\nabla_{\partial_i}\partial_j) - \nabla_{\partial_i}(\nabla_{\partial_\alpha}\partial_j), \partial_k\Big) \\
&= -g\Big(\nabla^{(1)}_{(\partial_\alpha)_1}\big((\nabla_{\partial_i}\partial_j)_1\big) + \nabla^{(2)}_{(\partial_\alpha)_2}\big((\nabla_{\partial_i}\partial_j)_2\big) - \nabla^{(1)}_{(\partial_i)_1}\big((\nabla_{\partial_\alpha}\partial_j)_1\big) - \nabla^{(2)}_{(\partial_i)_2}\big((\nabla_{\partial_\alpha}\partial_j)_2\big), \partial_k\Big) \\
&= -g\Big(0 + 0 - 0 - 0, \partial_k\Big) \\
&= 0
\end{aligned}$$

and, similarly for the rest of the mixed components:

$$R_{\alpha\beta ij} = R_{\alpha i\beta j} = R_{\alpha\beta\gamma i} = 0.$$

Therefore, (10) holds.

(b) In part (a) of this exercise, we computed that, in a product coordinate system $(x^1, \dots, x^n; y^1, \dots, y^m)$, the mixed components of the Riemann curvature tensor of the form $R_{\alpha i\beta j}$ vanish identically. Therefore, if $\Pi \subset T_p\mathcal{M}$ is the 2-plane spanned by the coordinate vector fields $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^\alpha}$, then

$$K_p(\Pi) = \frac{R_{i\alpha i\alpha}}{\|\partial_i \wedge \partial_\alpha\|^2} = 0.$$

Hence, the sectional curvature cannot be strictly positive or strictly negative for all 2-planes in $T_p\mathcal{M}$.

(c) Let S_1 be one of the equators of $(\mathbb{S}^2, g_{\mathbb{S}^2})$ (in the standard spherical coordinates (θ, ϕ) on $\mathbb{S}^2$, $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$, we can pick S_1 to be the curve $\theta = \frac{\pi}{2}$). Let $\gamma : [0, 2\pi) \to S_1$ be a geodesic parametrization of S_1 (so that $\gamma(t)$ is a geodesic curve in $(\mathbb{S}^2, g_{\mathbb{S}^2})$.

In the product Riemannian manifold $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^2})$, let us consider the 2-surface $S = S_1 \times S_1$; this surface is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^2$ and is parametrized by $\Psi : [0, 2\pi) \times [0, 2\pi) \to \mathbb{S}^2 \times \mathbb{S}^2$, $\Psi(t, s) = (\gamma(t), \gamma(s))$. Since the curve $\gamma(t)$ is a geodesic of $(\mathbb{S}^2, g_{\mathbb{S}^2})$, our remark below (12) imples that all the curves of the form $t \to \big(\gamma(\lambda_1 t + t_1), \gamma(\lambda_2 t + t_2)\big) \in S$ for $\lambda_1, \lambda_2 \neq 0$ are *geodesics* of $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^2})$. Notice that, for each $p \in S$, the set of curves of this type that pass through p span a dense subset of T_pS. As we showed in part (a) of Exercise 9.3, the second fundamental form $B(\cdot, \cdot)$ of $S \subset \mathbb{S}^2 \times \mathbb{S}^2$ must vanish in those directions; hence, since $B(\cdot, \cdot)$ is bilinear (and, therefore, continuous) on T_pS, we must have

$$B(v, v) = 0 \quad \text{for all } v \in T_pS.$$

Since $B(\cdot, \cdot)$ is also symmetric, we infer that

$$B(v, w) = 0 \quad \text{for all } v, w \in T_pS.$$

Therefore, S is a totally geodesic submanifold of $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^2})$.